

On the structures of split Leibniz triple systems

Yan Cao^{1,2}, Liangyun Chen¹

¹ School of Mathematics and Statistics, Northeast Normal University,
Changchun 130024, China

² Department of Basic Education, Harbin University of Science and Technology,
Rongcheng Campus, Rongcheng 264300, China

Abstract

We study the structures of arbitrary split Leibniz triple systems. By developing techniques of connections of roots for this kind of triple systems, under certain conditions, in the case of T being of maximal length, the simplicity of the Leibniz triple systems is characterized.

Key words: split Leibniz triple system, Lie triple system, Leibniz algebra, system of roots, root space

MSC(2010): 17A32, 17A60, 17B22, 17B65

1 Introduction

Leibniz triple systems were introduced by Bremner and Sánchez-Ortega [1]. Leibniz triple systems were defined in a functorial manner using the Kolesnikov-Pozhidaev algorithm, which took the defining identities for a variety of algebras and produced the defining identities for the corresponding variety of dialgebras [2]. In [1], Leibniz triple systems were obtained by applying the Kolesnikov-Pozhidaev algorithm to Lie triple systems. In [3], Levi's theorem for Leibniz triple systems is determined. Furthermore, Leibniz triple systems are related to Leibniz algebras in the same way that Lie triple systems related to Lie algebras. So it is natural to prove analogs of results from the theory of Lie triple systems to Leibniz triple systems.

Corresponding author(L. Chen): chenly640@nenu.edu.cn.

Supported by NNSF of China (Nos. 11171055 and 11471090), NSF of Jilin province (No. 201115006), Scientific Research Fund of Heilongjiang Provincial Education Department (No. 12541184).

In the present paper, we are interested in studying the structures of arbitrary Leibniz triple systems by focussing on the split ones. The class of the split ones is specially related to addition quantum numbers, graded contractions, and deformations. For instance, for a physical system which displays a symmetry of T , it is interesting to know in detail the structure of the split decomposition because its roots can be seen as certain eigenvalues which are the additive quantum numbers characterizing the state of such system. Recently, in [4–7], the structures of arbitrary split Lie algebras, arbitrary split Leibniz algebras and arbitrary split Lie triple systems have been determined by the techniques of connections of roots. Our work is essentially motivated by the work on split Leibniz algebras and split Lie triple systems [4, 5].

Throughout this paper, Leibniz triple systems T are considered of arbitrary dimension and over an arbitrary field \mathbb{K} . It is worth to mention that, unless otherwise stated, there is not any restriction on $\dim T_\alpha$ or $\{k \in \mathbb{K}: k\alpha \in \Lambda^1, \text{ for a fixed } \alpha \in \Lambda^1\}$, where T_α denotes the root space associated to the root α , and Λ^1 the set of nonzero roots of T . This paper proceeds as follows. In section 2, we establish the preliminaries on split Leibniz triple systems theory. In section 3, we show that under certain conditions, in the case of T being of maximal length, the simplicity of the Leibniz triple systems is characterized.

2 Preliminaries

Definition 2.1. [5] A **right Leibniz algebra** L is a vector space over a field \mathbb{K} endowed with a bilinear product $[\cdot, \cdot]$ satisfying the Leibniz identity

$$[[y, z], x] = [[y, x], z] + [y, [z, x]],$$

for all $x, y, z \in L$.

Definition 2.2. [1] A **Leibniz triple system** is a vector space T endowed with a trilinear operation $\{\cdot, \cdot, \cdot\} : T \times T \times T \rightarrow T$ satisfying

$$\{a, \{b, c, d\}, e\} = \{\{a, b, c\}, d, e\} - \{\{a, c, b\}, d, e\} - \{\{a, d, b\}, c, e\} + \{\{a, d, c\}, b, e\}, \quad (2.1)$$

$$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{\{a, b, d\}, c, e\} - \{\{a, b, e\}, c, d\} + \{\{a, b, e\}, d, c\}, \quad (2.2)$$

for all $a, b, c, d, e \in T$.

Example 2.3. A Lie triple system gives a Leibniz triple system with the same ternary product. If L is a Leibniz algebra with product $[\cdot, \cdot]$, then L becomes a Leibniz triple system by putting $\{x, y, z\} = [[x, y], z]$. More examples refer to [1].

Definition 2.4. [1] Let I be a subspace of a Leibniz triple system T . Then I is called a **subsystem** of T , if $\{I, I, I\} \subseteq I$; I is called an **ideal** of T , if $\{I, T, T\} + \{T, I, T\} + \{T, T, I\} \subseteq I$.

Definition 2.5. The **annihilator** of a Leibniz triple system T is the set $\text{Ann}(T) = \{x \in T : \{x, T, T\} + \{T, x, T\} + \{T, T, x\} = 0\}$.

Proposition 2.6. [3] *Let T be a Leibniz triple system. Then the following assertions hold.*

- (1) *J is generated by $\{\{a, b, c\} - \{a, c, b\} + \{b, c, a\} : a, b, c \in T\}$, then J is an ideal of T satisfying $\{T, T, J\} = \{T, J, T\} = 0$.*
- (2) *J is generated by $\{\{a, b, c\} - \{a, c, b\} + \{b, c, a\} : a, b, c \in T\}$, then T is a Lie triple system if and only if $J = 0$.*
- (3) *$\{\{c, d, e\}, b, a\} - \{\{c, d, e\}, a, b\} - \{\{c, b, a\}, d, e\} + \{\{c, a, b\}, d, e\} - \{c, \{a, b, d\}, e\} - \{c, d, \{a, b, e\}\} = 0$, for all $a, b, c, d, e \in T$.*

Definition 2.7. *A Leibniz triple system T is said to be **simple** if its product is nonzero and its only ideals are $\{0\}$, J and T , where J is generated by $\{\{a, b, c\} - \{a, c, b\} + \{b, c, a\} : a, b, c \in T\}$.*

It should be noted that the above definition agrees with the definition of a simple Lie triple system, since $J = \{0\}$ in this case.

Definition 2.8. [1] *The **standard embedding** of a Leibniz triple system T is the two-graded right Leibniz algebra $L = L^0 \oplus L^1$, L^0 being the \mathbb{K} -span of $\{x \otimes y, x, y \in T\}$, $L^1 := T$ and where the product is given by*

$$[(x \otimes y, z), (u \otimes v, w)] := (\{x, y, u\} \otimes v - \{x, y, v\} \otimes u + z \otimes w, \{x, y, w\} + \{z, u, v\} - \{z, v, u\}).$$

Let us observe that L^0 with the product induced by the one in $L = L^0 \oplus L^1$ becomes a right Leibniz algebra.

Definition 2.9. *Let T be a Leibniz triple system, $L = L^0 \oplus L^1$ be its standard embedding, and H^0 be a maximal abelian subalgebra (MASA) of L^0 . For a linear function $\alpha \in (H^0)^*$, we define the root space of T (with respect to H^0) associated to α as the subspace $T_\alpha := \{t_\alpha \in T : [t_\alpha, h] = \alpha(h)t_\alpha \text{ for any } h \in H^0\}$. The elements $\alpha \in (H^0)^*$ satisfying $T_\alpha \neq 0$ are called roots of T with respect to H^0 and we denote $\Lambda^1 := \{\alpha \in (H^0)^* \setminus \{0\} : T_\alpha \neq 0\}$.*

Let us observe that $T_0 = \{t_0 \in T : [t_0, h] = 0 \text{ for any } h \in H^0\}$. In the following, we shall denote by Λ^0 the set of all nonzero $\alpha \in (H^0)^*$ such that $L_\alpha^0 := \{v_\alpha^0 \in L^0 : [v_\alpha^0, h] = \alpha(h)v_\alpha^0 \text{ for any } h \in H^0\} \neq 0$.

Lemma 2.10. *Let T be a Leibniz triple system, $L = L^0 \oplus L^1$ be its standard embedding, and H^0 be a MASA of L^0 . For $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}$ and $\delta \in \Lambda^0 \cup \{0\}$, the following assertions hold.*

- (1) *If $[T_\alpha, T_\beta] \neq 0$ then $\alpha + \beta \in \Lambda^0 \cup \{0\}$ and $[T_\alpha, T_\beta] \subseteq L_{\alpha+\beta}^0$.*
- (2) *If $[L_\delta^0, T_\alpha] \neq 0$ then $\delta + \alpha \in \Lambda^1 \cup \{0\}$ and $[L_\delta^0, T_\alpha] \subseteq T_{\delta+\alpha}$.*
- (3) *If $[T_\alpha, L_\delta^0] \neq 0$ then $\alpha + \delta \in \Lambda^1 \cup \{0\}$ and $[T_\alpha, L_\delta^0] \subseteq T_{\alpha+\delta}$.*
- (4) *If $[L_\delta^0, L_\gamma^0] \neq 0$ then $\delta + \gamma \in \Lambda^0 \cup \{0\}$ and $[L_\delta^0, L_\gamma^0] \subseteq L_{\delta+\gamma}^0$.*
- (5) *If $\{T_\alpha, T_\beta, T_\gamma\} \neq 0$ then $\alpha + \beta + \gamma \in \Lambda^1 \cup \{0\}$ and $\{T_\alpha, T_\beta, T_\gamma\} \subseteq T_{\alpha+\beta+\gamma}$.*

Proof. (1) For any $x \in T_\alpha$, $y \in T_\beta$ and $h \in H^0$, by Leibniz identity, one has $[[x, y], h] = [x, [y, h]] + [[x, h], y] = [x, \beta(h)y] + [\alpha(h)x, y] = (\alpha + \beta)(h)[x, y]$.

(2) For any $x \in L_\delta^0$, $y \in T_\alpha$ and $h \in H^0$, by Leibniz identity, one has $[[x, y], h] = [x, [y, h]] + [[x, h], y] = [x, \alpha(h)y] + [\delta(h)x, y] = (\delta + \alpha)(h)[x, y]$.

(3) For any $x \in T_\alpha$, $y \in L_\delta^0$, and $h \in H^0$, by Leibniz identity, one has $[[x, y], h] = [x, [y, h]] + [[x, h], y] = [x, \delta(h)y] + [\alpha(h)x, y] = (\alpha + \delta)(h)[x, y]$.

(4) For any $x \in L_\delta^0$, $y \in L_\gamma^0$ and $h \in H^0$, by Leibniz identity, one has $[[x, y], h] = [x, [y, h]] + [[x, h], y] = [x, \gamma(h)y] + [\delta(h)x, y] = (\delta + \gamma)(h)[x, y]$.

(5) It is a consequence of Lemma 2.10 (1) and (2). \square

Definition 2.11. Let T be a Leibniz triple system, $L = L^0 \oplus L^1$ be its standard embedding, and H^0 be a MASA of L^0 . We shall call that T is a **split Leibniz triple system** (with respect to H^0) if :

- (1) $T = T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha)$,
- (2) $\{T_0, T_0, T_0\} = 0$,
- (3) $\{T_\alpha, T_{-\alpha}, T_0\} = 0$, for $\alpha \in \Lambda^1$.

We say that Λ^1 is the root system of T .

We also note that the facts $H^0 \subset L^0 = [T, T]$ and $T = T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha)$ imply

$$H^0 = [T_0, T_0] + \sum_{\alpha \in \Lambda^1} [T_\alpha, T_{-\alpha}]. \quad (2.3)$$

Finally, as $[T_0, T_0] \subset L_0^0 = H^0$, we have

$$[T_0, [T_0, T_0]] = 0. \quad (2.4)$$

We finally note that $\alpha \in \Lambda^1$ does not imply $\alpha \in \Lambda^0$.

Definition 2.12. A root system Λ^1 of a split Leibniz triple system T is called **symmetric** if it satisfies that $\alpha \in \Lambda^1$ implies $-\alpha \in \Lambda^1$.

A similar concept applies to the set Λ^0 of nonzero roots of L^0 .

In the following, T denotes a split Leibniz triple system with a symmetric root system Λ^1 , and $T = T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha)$ the corresponding root decomposition. We begin the study of split Leibniz triple systems by developing the concept of connections of roots.

Definition 2.13. Let α and β be two nonzero roots, we shall say that α and β are **connected** if there exists a family $\{\alpha_1, \alpha_2, \dots, \alpha_{2n}, \alpha_{2n+1}\} \subset \Lambda^1 \cup \{0\}$ of roots of T such that

- (1) $\{\alpha_1, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \dots, \alpha_1 + \dots + \alpha_{2n} + \alpha_{2n+1}\} \subset \Lambda^1$,
- (2) $\{\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \dots, \alpha_1 + \dots + \alpha_{2n}\} \subset \Lambda^0$,

(3) $\alpha_1 = \alpha$ and $\alpha_1 + \cdots + \alpha_{2n} + \alpha_{2n+1} = \pm\beta$.

We shall also say that $\{\alpha_1, \alpha_2, \dots, \alpha_{2n}, \alpha_{2n+1}\}$ is a connection from α to β .

We denote by

$$\Lambda_\alpha^1 := \{\beta \in \Lambda^1 : \alpha \text{ and } \beta \text{ are connected}\},$$

we can easily get that $\{\alpha\}$ is a connection from α to itself and to $-\alpha$. Therefore $\pm\alpha \in \Lambda_\alpha^1$.

Definition 2.14. A subset Ω^1 of a root system Λ^1 , associated to a split Leibniz triple system T , is called a **root subsystem** if it is symmetric, and for $\alpha, \beta, \gamma \in \Omega^1 \cup \{0\}$ such that $\alpha + \beta \in \Lambda^0$ and $\alpha + \beta + \gamma \in \Lambda^1$ then $\alpha + \beta + \gamma \in \Omega^1$.

Let Ω^1 be a root subsystem of Λ^1 . We define

$$T_{0,\Omega^1} := \text{span}_{\mathbb{K}}\{\{T_\alpha, T_\beta, T_\gamma\} : \alpha + \beta + \gamma = 0; \alpha, \beta, \gamma \in \Omega^1 \cup \{0\}\} \subset T_0$$

and $V_{\Omega^1} := \oplus_{\alpha \in \Omega^1} T_\alpha$. Taking into account the fact that $\{T_0, T_0, T_0\} = 0$, it is straightforward to verify that $T_{\Omega^1} := T_{0,\Omega^1} \oplus V_{\Omega^1}$ is a subsystem of T . We will say that T_{Ω^1} is a subsystem associated to the root subsystem Ω^1 .

Proposition 2.15. If Λ^0 is symmetric, then the relation \sim in Λ^1 , defined by $\alpha \sim \beta$ if and only if $\beta \in \Lambda_\alpha^1$, is of equivalence.

Proof. This can be proved completely analogously to [8, Proposition 3.1]. \square

Proposition 2.16. Let α be a nonzero root and suppose Λ^0 is symmetric. Then Λ_α^1 is a root subsystem.

Proof. This can be proved completely analogously to [8, Lemma 3.1]. \square

3 Split Leibniz triple system of maximal length. The simple case.

In this section we focus on the simplicity of split Leibniz triple systems by centering our attention in those of maximal length. From now on $\text{char}(\mathbb{K})=0$.

Definition 3.1. We say that a split Leibniz triple systems T is of **maximal length** if $\dim T_\alpha = 1$ for any $\alpha \in \Lambda^1$.

Lemma 3.2. Let T be a split Leibniz triple systems. For any $\alpha, \beta \in \Lambda^1$ with $\alpha \neq k\beta$, $k \in \mathbb{K}$, there exists $h_{\alpha,\beta} \in H^0$ such that $\alpha(h_{\alpha,\beta}) \neq 0$ and $\beta(h_{\alpha,\beta}) = 0$.

Proof. As $\alpha \neq 0$, there exists $h \in H^0 - \{0\}$ such that $\alpha(h) \neq 0$. If $\beta(h) = 0$ we take $h_{\alpha,\beta} := h$. Suppose therefore $\beta(h) \neq 0$, let us write $k = \frac{\alpha(h)}{\beta(h)}$. As $\alpha \neq k\beta$, there exists $h' \in H^0$ such that $\alpha(h') \neq k\beta(h')$. we can take $h_{\alpha,\beta} := \beta(h')h - \beta(h)h'$. \square

Lemma 3.3. *Let $T = T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha)$ be a split Leibniz triple systems. If I is an ideal of T then $I = (I \cap T_0) \oplus (\oplus_{\alpha \in \Lambda^1} (I \cap T_\alpha))$.*

Proof. Let $x \in I$. We can write $x = t_0 + \sum_{j=1}^m e_{\beta_j}$, for $t_0 \in T_0$, $e_{\beta_j} \in T_{\beta_j}$ and $\beta_j \neq \beta_k$ if $j \neq k$. Let us show that any $e_{\beta_j} \in I$. If $e_{\beta_1} = 0$ then $e_{\beta_1} \in I$. Suppose $e_{\beta_1} \neq 0$. For any $\beta_{k_r} \neq p\beta_1, p \in \mathbb{K}$ and $k_r \in \{2, \dots, m\}$, Lemma 3.2 gives us $h_{\beta_1, \beta_{k_r}} \in H^0$ satisfying $\beta_1(h_{\beta_1, \beta_{k_r}}) \neq 0$ and $\beta_{k_r}(h_{\beta_1, \beta_{k_r}}) = 0$. From here,

$$[[\dots [x, h_{\beta_1, \beta_{k_2}}], h_{\beta_1, \beta_{k_3}}], \dots], h_{\beta_1, \beta_{k_s}}] = p_1 e_{\beta_1} + \sum_{t=1}^u p_{k_t} e_{k_t \beta_1} \in I, \quad (3.5)$$

where $p_1, k_t \in \mathbb{K} - \{0\}$, $k_t \neq 1$ and $p_{k_t} \in \mathbb{K}$.

If any $p_{k_t} = 0, t = 1, \dots, u$, then $p_1 e_{\beta_1} \in I$ and so $e_{\beta_1} \in I$. Let us suppose some $p_{k_t} \neq 0$ and write (3.5) as

$$p_1 e_{\beta_1} + \sum_{t=1}^v p_{k_t} e_{k_t \beta_1} \in I, \quad (3.6)$$

where $p_1, k_t, p_{k_t} \in \mathbb{K} - \{0\}$, $k_t \neq 1, v \leq u$.

Let $h \in H^0$ such that $\beta_1(h) \neq 0$. Then

$$[p_1 e_{\beta_1} + \sum_{t=1}^v p_{k_t} e_{k_t \beta_1}, h] = p_1 \beta_1(h) e_{\beta_1} + \sum_{t=1}^v p_{k_t} k_t \beta_1(h) e_{k_t \beta_1} \in I,$$

and so

$$p_1 e_{\beta_1} + \sum_{t=1}^v p_{k_t} k_t e_{k_t \beta_1} \in I, \quad k_t \neq 1. \quad (3.7)$$

From (3.6) and (3.7), it follows easily that

$$q_1 e_{\beta_1} + \sum_{t=1}^w q_{k_t} e_{q_t \beta_1} \in I, \quad (3.8)$$

where $q_1, q_{k_t} \in \mathbb{K} - \{0\}$, $q_t \in \{k_t : t = 1, \dots, v\}$ and $w < v$.

Following this process (multiply (3.8) with h and compare the result with (3.8) taking into account $q_t \neq 1$, and so on), we obtain $e_{\beta_1} \in I$. The same argument holds for any $\beta_j, j \neq 1$. From here, we deduce $I = (I \cap T_0) \oplus (\oplus_{\alpha \in \Lambda^1} (I \cap T_\alpha))$. \square

Let us return to a split Leibniz triple system of maximal length T . From now on $T = T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha)$ denotes a split Leibniz triple system of maximal length. Using the previous Lemma, we assert that given any nonzero ideal I of T then

$$I = (I \cap T_0) \oplus (\oplus_{\alpha \in \Lambda^I} T_\alpha), \quad (3.9)$$

where $\Lambda^I := \{\alpha \in \Lambda^1 : I \cap T_\alpha \neq 0\}$.

In particular, case $I=J$, we get

$$J = (J \cap T_0) \oplus (\oplus_{\alpha \in \Lambda^J} T_\alpha). \quad (3.10)$$

From here, we can write

$$\Lambda^1 = \Lambda^J \cup \Lambda^{-J}, \quad (3.11)$$

where

$$\Lambda^J := \{\alpha \in \Lambda^1 : T_\alpha \subset J\}$$

and

$$\Lambda^{-J} := \{\alpha \in \Lambda^1 : T_\alpha \cap J = 0\}.$$

As consequence

$$T = T_0 \oplus (\oplus_{\alpha \in \Lambda^{-J}} T_\alpha) \oplus (\oplus_{\beta \in \Lambda^J} T_\beta). \quad (3.12)$$

Next, we will consider T satisfying $\{T_\alpha, T_0, T_\beta\} \neq 0$, for $\alpha \in \Lambda^J$ and $\beta \in \Lambda^{-J}$. Under this assumption, the fact that $T = \{T, T, T\}$, the split decomposition given by (3.12) and the Definition of a split Leibniz triple system show

$$T_0 = \sum_{\substack{\alpha, \beta, \gamma \in \Lambda^{-J} \cup \{0\} \\ \alpha + \beta + \gamma = 0}} \{T_\alpha, T_\beta, T_\gamma\}. \quad (3.13)$$

Now, observe that the concept of connectivity of nonzero roots given in Definition 2.13 is not strong enough to detect if a given $\alpha \in \Lambda^1$ belongs to Λ^J or to Λ^{-J} . Consequently we lose the information respect to whether a given root space T_α is contained in J or not, which is fundamental to study the simplicity of T . So, we are going to refine the concept of connections of nonzero roots in the setup of maximal length split Leibniz triple systems. The symmetry of Λ^J and Λ^{-J} will be understood as usual. That is, $\Lambda^\gamma, \gamma \in \{J, \neg J\}$, is called symmetric if $\alpha \in \Lambda^\gamma$ implies $-\alpha \in \Lambda^\gamma$.

In the following, T denotes a split Leibniz triple system whose root space satisfies $\{T_\alpha, T_0, T_\beta\} \neq 0$, where $\alpha \in \Lambda^J$ and $\beta \in \Lambda^{-J}$.

Definition 3.4. Let $\alpha, \beta \in \Lambda^\gamma$ with $\gamma \in \{J, \neg J\}$. We say that α is $\neg J$ -connected to β , denoted by $\alpha \sim_{\neg J} \beta$, if there exist

$$\alpha_2, \dots, \alpha_{2n+1} \in \Lambda^{-J} \cup \{0\}$$

such that

- (1) $\{\alpha_1, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \dots, \alpha_1 + \dots + \alpha_{2n} + \alpha_{2n+1}\} \subset \Lambda^\gamma$,
- (2) $\{\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \dots, \alpha_1 + \dots + \alpha_{2n}\} \subset \Lambda^0$,
- (3) $\alpha_1 = \alpha$ and $\alpha_1 + \dots + \alpha_{2n} + \alpha_{2n+1} = \pm \beta$.

We shall also say that $\{\alpha_1, \alpha_2, \dots, \alpha_{2n}, \alpha_{2n+1}\}$ is a $\neg J$ -connection from α to β .

Proposition 3.5. *The following assertions hold.*

- (1) *If $\Lambda^{\neg J}$ is symmetric, then the relation $\sim_{\neg J}$ is an equivalence relation in $\Lambda^{\neg J}$.*
- (2) *If $T = \{T, T, T\}$ and $\Lambda^{\neg J}$, Λ^J are symmetric, then the relation $\sim_{\neg J}$ is an equivalence relation in Λ^J .*

Proof. (1) Can be proved in a similar way to Proposition 2.15.

(2) Note that $T_0 = \sum_{\substack{\alpha, \beta, \gamma \in \Lambda^{\neg J} \cup \{0\} \\ \alpha + \beta + \gamma = 0}} \{T_\alpha, T_\beta, T_\gamma\}$ and $H^0 = [T_0, T_0] + \sum_{\alpha \in \Lambda^{\neg J}} [T_\alpha, T_{-\alpha}] + \sum_{\alpha \in \Lambda^J} [T_\alpha, T_{-\alpha}]$. Let $\delta \in \Lambda^J$, one gets

$$[T_\delta, \sum_{\alpha \in \Lambda^J} [T_\alpha, T_{-\alpha}]] \subset \sum_{\alpha \in \Lambda^J} [[T_\delta, T_\alpha], T_{-\alpha}] + \sum_{\alpha \in \Lambda^J} [[T_\delta, T_{-\alpha}], T_\alpha] = 0.$$

Since $\delta \neq 0$, one gets either $[T_\delta, [T_0, T_0]] \neq 0$ or $[T_\delta, \sum_{\alpha \in \Lambda^{\neg J}} [T_\alpha, T_{-\alpha}]] \neq 0$.

Suppose $[T_\delta, [T_0, T_0]] \neq 0$. By Leibniz identity, one gets $[[T_\delta, T_0], T_0] \neq 0$. Then $\neg J$ -connection $\{\delta, 0, 0\}$ gives us $\delta \sim_{\neg J} \delta$.

Suppose $[T_\delta, \sum_{\alpha \in \Lambda^{\neg J}} [T_\alpha, T_{-\alpha}]] \neq 0$. By Leibniz identity, either $[[T_\delta, T_\alpha], T_{-\alpha}] \neq 0$ or $[[T_\delta, T_{-\alpha}], T_\alpha] \neq 0$. In the first case, the $\neg J$ -connection $\{\delta, \alpha, -\alpha\}$ gives us $\delta \sim_{\neg J} \delta$ while in the second case the $\neg J$ -connection $\{\delta, -\alpha, \alpha\}$ gives us the same result.

Consequently, $\sim_{\neg J}$ is reflexive in Λ^J . The symmetric and transitive character of $\sim_{\neg J}$ in Λ^J follows as in Proposition 2.15. \square

Let us introduce the notion of root-multiplicativity in the framework of split Leibniz triple systems of maximal length, in a similar way to the ones for split Leibniz algebras and split Lie triple systems (see [4, 5] for these notions and examples).

Definition 3.6. *We say that a split Leibniz triple system of maximal length T is root-multiplicative if the below conditions hold.*

- (1) *Given $\alpha, \beta, \gamma \in \Lambda^{\neg J} \cup \{0\}$ such that $\alpha + \beta \in \Lambda^0$ and $\alpha + \beta + \gamma \in \Lambda^1$, then $\{T_\alpha, T_\beta, T_\gamma\} \neq 0$.*
- (2) *Given $\alpha, \beta \in \Lambda^{\neg J} \cup \{0\}$ and $\gamma \in \Lambda^J$ such that $\alpha + \beta \in \Lambda^0$ and $\alpha + \beta + \gamma \in \Lambda^J$, then $\{T_\gamma, T_\beta, T_\alpha\} \neq 0$.*

Lemma 3.7. *Let T be a root-multiplicative split Leibniz triple systems with $\text{Ann}(T) = 0$. If for any $\alpha \in \Lambda^1$, we have $\dim L_\alpha^0 = 1$. Then there is not any nonzero ideal of T contained in T_0 .*

Proof. Suppose there exists a nonzero ideal of T such that $I \subset T_0$. The facts that $\{T_0, T_0, T_0\} = 0$ gives $\{I, T_0, T_0\} = 0$, $\{T_0, I, T_0\} = 0$ and $\{T_0, T_0, I\} = 0$. Given $\alpha \in \Lambda^1$, since

$$\begin{aligned} \{I, T_0, T_\alpha\} + \{I, T_\alpha, T_0\} &\subset T_\alpha \cap T_0 = 0, \\ \{T_0, I, T_\alpha\} + \{T_\alpha, I, T_0\} &\subset T_\alpha \cap T_0 = 0, \end{aligned}$$

and

$$\{T_0, T_\alpha, I\} + \{T_\alpha, T_0, I\} \subset T_\alpha \cap T_0 = 0,$$

one gets

$$\{I, T_0, T_\alpha\} = \{I, T_\alpha, T_0\} = 0,$$

$$\{T_0, I, T_\alpha\} = \{T_\alpha, I, T_0\} = 0,$$

and

$$\{T_0, T_\alpha, I\} = \{T_\alpha, T_0, I\} = 0.$$

Given also $\beta \in \Lambda^1$, with $\alpha + \beta \neq 0$, one has

$$\{I, T_\alpha, T_\beta\} \subset T_{\alpha+\beta} \cap T_0 = 0,$$

$$\{T_\alpha, I, T_\beta\} \subset T_{\alpha+\beta} \cap T_0 = 0,$$

$$\{T_\alpha, T_\beta, I\} \subset T_{\alpha+\beta} \cap T_0 = 0.$$

We also have $\{T_\alpha, T_{-\alpha}, I\} \subset \{T_\alpha, T_{-\alpha}, T_0\} = 0$ (see the Definition of a split Leibniz triple system). As $\text{Ann}(T) = 0$, one gets

$$\{I, T_\alpha, T_{-\alpha}\} \neq 0$$

or

$$\{T_\alpha, I, T_{-\alpha}\} \neq 0.$$

We treat separately two cases.

Case 1: $\{I, T_\alpha, T_{-\alpha}\} \neq 0$. Thus, there exist $t_{\pm\alpha} \in T_{\pm\alpha}$ and $t_0 \in I$ such that $\{t_0, t_\alpha, t_{-\alpha}\} \neq 0$. Hence, $0 \neq [t_0, t_\alpha] \in L_\alpha^0$. Using $\dim L_\alpha^0 = 1$ and the root-multiplicativity of T (consider the roots $0, \alpha, 0 \in \Lambda^1 \cup \{0\}$), there exists $0 \neq t'_0 \in T_0$ such that $0 \neq \{t_0, t_\alpha, t'_0\} \in T_\alpha$. As $t_0 \in I$, we conclude $0 \neq t'_\alpha := \{t_0, t_\alpha, t'_0\} \in I \subset T_0$, a contradiction. Hence I is not contained in T_0 .

Case 2: $\{T_\alpha, I, T_{-\alpha}\} \neq 0$. Thus, there exist $t_{\pm\alpha} \in T_{\pm\alpha}$ and $t_0 \in I$ such that $\{t_\alpha, t_0, t_{-\alpha}\} \neq 0$. Hence, $0 \neq [t_\alpha, t_0] \in L_\alpha^0$. Using $\dim L_\alpha^0 = 1$ and the root-multiplicativity of T (consider the roots $\alpha, 0, 0 \in \Lambda^1 \cup \{0\}$), there exists $0 \neq t'_0 \in T_0$ such that $0 \neq \{t_\alpha, t_0, t'_0\} \in T_\alpha$. As $t_0 \in I$, we conclude $0 \neq t'_\alpha := \{t_\alpha, t_0, t'_0\} \in I \subset T_0$, a contradiction. Hence I is not contained in T_0 . \square

Another interesting notion related to split Leibniz triple systems of maximal length T is those of Lie-annihilator. Write $T = T_0 \oplus (\oplus_{\alpha \in \Lambda^{-J}} T_\alpha) \oplus (\oplus_{\beta \in \Lambda^J} T_\beta)$ (see (3.12)).

Definition 3.8. *The Lie-annihilator of a split Leibniz triple system of maximal length T is the set*

$$\begin{aligned} \text{Ann}_{\text{Lie}}(T) = & \left\{ x \in T : \{x, T_0 \oplus (\oplus_{\alpha \in \Lambda^{-J}} T_\alpha), T_0 \oplus (\oplus_{\alpha \in \Lambda^{-J}} T_\alpha)\} \right. \\ & + \{T_0 \oplus (\oplus_{\alpha \in \Lambda^{-J}} T_\alpha), x, T_0 \oplus (\oplus_{\alpha \in \Lambda^{-J}} T_\alpha)\} \\ & \left. + \{T_0 \oplus (\oplus_{\alpha \in \Lambda^{-J}} T_\alpha), T_0 \oplus (\oplus_{\alpha \in \Lambda^{-J}} T_\alpha), x\} = 0 \right\}. \end{aligned}$$

Clearly the above Definition agrees with the Definition of annihilator of a Lie triple system, since in this case $\Lambda^J = \emptyset$. We also have $\text{Ann}(T) \subset \text{Ann}_{\text{Lie}}(T)$.

Proposition 3.9. *Suppose $T = \{T, T, T\}$ and T is root-multiplicative. If $\Lambda^{\neg J}$ has all of its roots $\neg J$ -connected, then any ideal I of T such that $I \not\subseteq T_0 \oplus J$ satisfies $I = T$.*

Proof. By (3.9) and (3.11), we can write

$$I = (I \cap T_0) \oplus (\oplus_{\alpha_i \in \Lambda^{\neg J, I}} T_{\alpha_i}) \oplus (\oplus_{\beta_j \in \Lambda^{J, I}} T_{\beta_j}),$$

where $\Lambda^{\neg J, I} := \Lambda^{\neg J} \cap \Lambda^I$ and $\Lambda^{J, I} := \Lambda^J \cap \Lambda^I$. As $I \not\subseteq T_0 \oplus J$, one gets $\Lambda^{\neg J, I} \neq \emptyset$ and so we can fix some $\gamma_0 \in \Lambda^{\neg J, I}$ such that

$$T_{\gamma_0} \subset I. \quad (3.14)$$

For any $\beta \in \Lambda^{\neg J}$, $\beta \neq \pm\gamma_0$. The fact that γ_0 and β are $\neg J$ -connected gives us a $\neg J$ -connection $\{\gamma_1, \dots, \gamma_{2r+1}\} \subset \Lambda^{\neg J} \cup \{0\}$ from γ_0 to β such that

$$\gamma_1 = \gamma_0,$$

$$\gamma_1 + \gamma_2, \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, \dots, \gamma_1 + \dots + \gamma_{2r} \in \Lambda^0,$$

$$\gamma_1, \gamma_1 + \gamma_2 + \gamma_3, \dots, \gamma_1 + \dots + \gamma_{2r} + \gamma_{2r+1} \in \Lambda^{\neg J},$$

and $\gamma_1 + \dots + \gamma_{2r} + \gamma_{2r+1} = \pm\beta$.

Consider $\gamma_0 = \gamma_1, \gamma_2, \gamma_3$ and $\gamma_1 + \gamma_2 + \gamma_3$. Since $\gamma_1, \gamma_2, \gamma_3 \in \Lambda^{\neg J} \cup \{0\}$, the root-multiplicativity and maximal length of T show $\{T_{\gamma_0}, T_{\gamma_2}, T_{\gamma_3}\} = T_{\gamma_0+\gamma_2+\gamma_3}$, and by (3.14),

$$T_{\gamma_0+\gamma_2+\gamma_3} \subset I.$$

Following this process with the $\neg J$ -connection $\{\gamma_1, \dots, \gamma_{2r+1}\}$, we obtain that

$$T_{\gamma_0+\gamma_2+\gamma_3+\dots+\gamma_{2r+1}} \subset I.$$

From here, we get that either

$$T_\beta \subset I \text{ or } T_{-\beta} \subset I, \quad (3.15)$$

for any $\beta \in \Lambda^{\neg J}$.

Observe that as a consequence of $T = \{T, T, T\}$ and $\{T_\alpha, T_0, T_\beta\} \neq 0$ where $\alpha \in \Lambda^J$, $\beta \in \Lambda^{\neg J}$, one gets

$$T_0 = \sum_{\substack{\alpha, \beta, \gamma \in \Lambda^{\neg J} \cup \{0\} \\ \alpha + \beta + \gamma = 0}} \{T_\alpha, T_\beta, T_\gamma\}. \quad (3.16)$$

Let us study the products $\{T_\alpha, T_\beta, T_\gamma\}$ of (3.16) in order to show $T_0 \subset I$. Taking into account Definition 2.11 (2), (3) and the fact that $\alpha + \beta + \gamma = 0$ with $\alpha, \beta, \gamma \in \Lambda^{\neg J} \cup \{0\}$, we can suppose $\gamma \neq 0$ and either $\alpha \neq 0$ or $\beta \neq 0$. Suppose $\alpha \neq 0$ and $\beta = 0$ (resp. $\alpha = 0$ and $\beta \neq 0$), one gets $\alpha = -\gamma$ (resp. $\beta = -\gamma$), and by (3.15), $\{T_\alpha, T_\beta, T_\gamma\} = \{T_{-\gamma}, T_0, T_\gamma\} \subset I$, (resp. $\{T_\alpha, T_\beta, T_\gamma\} = \{T_0, T_{-\gamma}, T_\gamma\} \subset I$). If the three elements in $\{\alpha, \beta, \gamma\}$ are nonzero, in case some $T_\epsilon \subset I$, $\epsilon \in \{\alpha, \beta, \gamma\}$, then clearly $\{T_\alpha, T_\beta, T_\gamma\} \subset I$.

Finally, consider the case in which any of the T_ϵ does not belong to I . If $\{T_\alpha, T_\beta, T_\gamma\} = 0$ then $\{T_\alpha, T_\beta, T_\gamma\} \subset I$. If $\{T_\alpha, T_\beta, T_\gamma\} \neq 0$, necessarily $\alpha + \beta \neq 0$ and so $\alpha + \beta \in \Lambda^0$. From here, by root-multiplicativity, one gets $0 \neq \{T_\alpha, T_\beta, T_{-\beta}\} = T_\alpha$. By (3.15), we have $T_\alpha \subset I$ and so $\{T_\alpha, T_\beta, T_\gamma\} \subset I$. Therefore (3.16) implies

$$T_0 \subset I. \quad (3.17)$$

Now, given any $\delta \in \Lambda^1$, the facts $\delta \neq 0$, $T_0 \subset I$, root-multiplicativity and $H^0 = [T_0, T_0] + \sum_{\alpha \in \Lambda^{-J}} [T_\alpha, T_{-\alpha}] + \sum_{\alpha \in \Lambda^J} [T_\alpha, T_{-\alpha}]$ show either $[T_\delta, [T_0, T_0]] \neq 0$ or $[T_\delta, [T_\alpha, T_{-\alpha}]] \neq 0$, for $\alpha \in \Lambda^{-J}$. We will distinguish respectively.

Suppose $[T_\delta, [T_0, T_0]] \neq 0$. By Leibniz identity, we have $[T_\delta, [T_0, T_0]] \subseteq [[T_\delta, T_0], T_0]$. Since $T_0 \subset I$, one gets $[[T_\delta, T_0], T_0] \subset I$, and so $[T_\delta, [T_0, T_0]] \subset I$. By root-multiplicativity, $0 \neq [T_\delta, [T_0, T_0]] = T_\delta$, so

$$T_\delta \subset I, \quad (3.18)$$

for $\delta \in \Lambda^1$.

Suppose $[T_\delta, [T_\alpha, T_{-\alpha}]] \neq 0$, for $\alpha \in \Lambda^{-J}$. By Leibniz identity, we have $[T_\delta, [T_\alpha, T_{-\alpha}]] \subseteq [[T_\delta, T_\alpha], T_{-\alpha}] + [[T_\delta, T_{-\alpha}], T_\alpha]$. By (3.15), one gets $[[T_\delta, T_\alpha], T_{-\alpha}] \subset I$ and $[[T_\delta, T_{-\alpha}], T_\alpha] \subset I$ for $\alpha \in \Lambda^{-J}$. By root-multiplicativity, one gets $0 \neq [T_\delta, [T_\alpha, T_{-\alpha}]] = T_\delta$, and so

$$T_\delta \subset I, \quad (3.19)$$

for $\delta \in \Lambda^1$.

From (3.14), (3.18) and (3.19), we conclude $I = T$. \square

Proposition 3.10. *Suppose $T = \{T, T, T\}$, $\text{Ann}(T) = 0$ and T is root-multiplicative. If Λ^{-J} , Λ^J are symmetric, Λ^J has all of its roots $\neg J$ -connected and $\{T_0, T_\alpha, T_\beta\} = 0$, for $\alpha, \beta \in \Lambda^{-J}$, then any nonzero ideal I of T such that $I \subseteq J$ satisfies either $I = J$ or $J = I \oplus K$ with K an ideal of T .*

Proof. By (3.9) and (3.11), we can write

$$I = (I \cap T_0) \oplus (\oplus_{\alpha_i \in \Lambda^{J,I}} T_{\alpha_i}),$$

where $\Lambda^{J,I} \subset \Lambda^J$. Observe that the fact $\text{Ann}(T) = 0$ implies

$$J \cap T_0 = 0. \quad (3.20)$$

Indeed, from the fact that $\{T_0, T_0, T_0\} = 0$, one gets

$$\{J \cap T_0, T_0, T_0\} + \{T_0, J \cap T_0, T_0\} + \{T_0, T_0, J \cap T_0\} = 0. \quad (3.21)$$

By Proposition 2.6 (1), it is easy to see

$$\{J \cap T_0, T_0, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha\} + \{T_0, J \cap T_0, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha\} + \{T_0, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha, J \cap T_0\} = 0. \quad (3.22)$$

By Proposition 2.6 (1), it is easy to see

$$\{T_0, J \cap T_0, \oplus_{\beta \in \Lambda^{-J}} T_\beta\} = 0$$

and

$$\{T_0, \oplus_{\beta \in \Lambda^{-J}} T_\beta, J \cap T_0\} = 0.$$

As $\{J \cap T_0, T_0, \oplus_{\beta \in \Lambda^{-J}} T_\beta\} \subset \oplus_{\beta \in \Lambda^{-J}} T_\beta \cap J = 0$, one gets

$$\{J \cap T_0, T_0, \oplus_{\beta \in \Lambda^{-J}} T_\beta\} = 0.$$

Therefore, one gets

$$\{J \cap T_0, T_0, \oplus_{\beta \in \Lambda^{-J}} T_\beta\} + \{T_0, J \cap T_0, \oplus_{\beta \in \Lambda^{-J}} T_\beta\} + \{T_0, \oplus_{\beta \in \Lambda^{-J}} T_\beta, J \cap T_0\} = 0. \quad (3.23)$$

By Proposition 2.6 (1), it is easy to see

$$\{J \cap T_0, \oplus_{\alpha \in \Lambda^J} T_\alpha, T_0\} + \{\oplus_{\alpha \in \Lambda^J} T_\alpha, J \cap T_0, T_0\} + \{\oplus_{\alpha \in \Lambda^J} T_\alpha, T_0, J \cap T_0\} = 0. \quad (3.24)$$

Similarly, we also get

$$\{J \cap T_0, \oplus_{\alpha \in \Lambda^J} T_\alpha, \oplus_{\beta \in \Lambda^J} T_\beta\} = 0, \quad (3.25)$$

$$\{\oplus_{\alpha \in \Lambda^J} T_\alpha, J \cap T_0, \oplus_{\beta \in \Lambda^J} T_\beta\} = 0, \quad (3.26)$$

$$\{\oplus_{\alpha \in \Lambda^J} T_\alpha, \oplus_{\beta \in \Lambda^J} T_\beta, J \cap T_0\} = 0, \quad (3.27)$$

$$\{J \cap T_0, \oplus_{\alpha \in \Lambda^J} T_\alpha, \oplus_{\beta \in \Lambda^{-J}} T_\beta\} = 0, \quad (3.28)$$

$$\{\oplus_{\alpha \in \Lambda^J} T_\alpha, J \cap T_0, \oplus_{\beta \in \Lambda^{-J}} T_\beta\} = 0, \quad (3.29)$$

$$\{\oplus_{\alpha \in \Lambda^J} T_\alpha, \oplus_{\beta \in \Lambda^{-J}} T_\beta, J \cap T_0\} = 0, \quad (3.30)$$

$$\{J \cap T_0, \oplus_{\beta \in \Lambda^{-J}} T_\beta, T_0\} + \{\oplus_{\beta \in \Lambda^{-J}} T_\beta, J \cap T_0, T_0\} + \{\oplus_{\beta \in \Lambda^{-J}} T_\beta, T_0, J \cap T_0\} = 0, \quad (3.31)$$

$$\{J \cap T_0, \oplus_{\beta \in \Lambda^{-J}} T_\beta, \oplus_{\alpha \in \Lambda^J} T_\alpha\} = 0, \quad (3.32)$$

$$\{\oplus_{\beta \in \Lambda^{-J}} T_\beta, J \cap T_0, \oplus_{\alpha \in \Lambda^J} T_\alpha\} = 0, \quad (3.33)$$

$$\{\oplus_{\beta \in \Lambda^{\neg J}} T_\beta, \oplus_{\alpha \in \Lambda^J} T_\alpha, J \cap T_0\} = 0. \quad (3.34)$$

By Proposition 2.6 (1), it is easy to see

$$\{\oplus_{\beta \in \Lambda^{\neg J}} T_\beta, J \cap T_0, \oplus_{\alpha \in \Lambda^{\neg J}} T_\alpha\} = 0, \quad (3.35)$$

and

$$\{\oplus_{\beta \in \Lambda^{\neg J}} T_\beta, \oplus_{\alpha \in \Lambda^{\neg J}} T_\alpha, J \cap T_0\} = 0. \quad (3.36)$$

By known condition, we have

$$\{J \cap T_0, \oplus_{\beta \in \Lambda^{\neg J}} T_\beta, \oplus_{\alpha \in \Lambda^{\neg J}} T_\alpha\} = 0. \quad (3.37)$$

From (3.21), (3.22), (3.23), (3.24), (3.25), (3.26), (3.27), (3.28), (3.29), (3.30), (3.31), (3.31), (3.32), (3.33), (3.34), (3.35), (3.36), (3.37) and (3.12), one gets

$$\{J \cap T_0, T, T\} + \{T, J \cap T_0, T\} + \{T, T, J \cap T_0\} = 0.$$

From here $J \cap T_0 \subset \text{Ann}(T) = 0$. Hence, we can write

$$I = \oplus_{\alpha_i \in \Lambda^{J,I}} T_{\alpha_i},$$

with $\Lambda^{J,I} \neq \emptyset$, and so we can take some $\alpha_0 \in \Lambda^{J,I}$ such that $T_{\alpha_0} \subset I$. We can argue with the root-multiplicativity and the maximal length of T as in Proposition 3.9 to conclude that given any $\beta \in \Lambda^J$, there exists a $\neg J$ -connection $\{\gamma_1, \dots, \gamma_{2r+1}\}$ from α_0 to β such that

$$\{\{\dots, \{\{T_{\alpha_0}, T_{\gamma_2}, T_{\gamma_3}\}, T_{\gamma_4}, T_{\gamma_5}\}, \dots\}, T_{\gamma_{2r}}, T_{\gamma_{2r+1}}\} = T_{\pm\beta}$$

and so $T_{\epsilon\beta} \subset I$ for some $\epsilon \in \pm 1$. That is

$$\epsilon_\beta \beta \in \Lambda^{J,I} \text{ for any } \beta \in \Lambda^J \text{ and some } \epsilon_\beta \in \pm 1. \quad (3.38)$$

Suppose $-\alpha_0 \in \Lambda^{J,I}$. Then we also have that $\{-\gamma_1, \dots, -\gamma_{2r+1}\}$ from α_0 to β is a $\neg J$ -connection from $-\alpha_0$ to β satisfying

$$\{\{\dots, \{\{T_{-\alpha_0}, T_{-\gamma_2}, T_{-\gamma_3}\}, T_{-\gamma_4}, T_{-\gamma_5}\}, \dots\}, T_{-\gamma_{2r}}, T_{-\gamma_{2r+1}}\} = T_{-\epsilon_\beta \beta} \subset I$$

and so $T_\beta + T_{-\beta} \subset I$. Hence, (3.10) and (3.20) imply that $I = J$.

Now, suppose there is not any $\alpha_0 \in \Lambda^{J,I}$ such that $-\alpha_0 \in \Lambda^{J,I}$. (3.38) allows us to write $\Lambda^J = \Lambda^{J,I} \cup (-\Lambda^{J,I})$ and (together with (3.10) and (3.20)) asserts that by denoting $K = \oplus_{\alpha_i \in \Lambda^{J,I}} T_{-\alpha_i}$, we have

$$J = I \oplus K.$$

Let us finally show that K is an ideal of T . We have $\{T, K, T\} + \{T, T, K\} = 0$ and

$$\{K, T, T\}$$

$$\begin{aligned}
&= \{K, T_0 \oplus (\oplus_{\alpha \in \Lambda^{-J}} T_\alpha) \oplus (\oplus_{\beta \in \Lambda^J} T_\beta), T_0 \oplus (\oplus_{\alpha \in \Lambda^{-J}} T_\alpha) \oplus (\oplus_{\beta \in \Lambda^J} T_\beta)\} \\
&= \{K, T_0, T_0\} + \{K, T_0, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha\} + \{K, T_0, \oplus_{\beta \in \Lambda^J} T_\beta\} \\
&\quad + \{K, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha, T_0\} + \{K, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha, \oplus_{\gamma \in \Lambda^{-J}} T_\gamma\} + \{K, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha, \oplus_{\beta \in \Lambda^J} T_\beta\} \\
&\quad + \{K, \oplus_{\beta \in \Lambda^J} T_\beta, T_0\} + \{K, \oplus_{\beta \in \Lambda^J} T_\beta, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha\} + \{K, \oplus_{\beta \in \Lambda^J} T_\beta, \oplus_{\gamma \in \Lambda^J} T_\gamma\}.
\end{aligned}$$

Here, it is easy to see

$$\begin{aligned}
\{K, T_0, \oplus_{\beta \in \Lambda^J} T_\beta\} &= 0, \\
\{K, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha, \oplus_{\beta \in \Lambda^J} T_\beta\} &= 0, \\
\{K, \oplus_{\beta \in \Lambda^J} T_\beta, T_0\} &= 0, \\
\{K, \oplus_{\beta \in \Lambda^J} T_\beta, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha\} &= 0, \\
\{K, \oplus_{\beta \in \Lambda^J} T_\beta, \oplus_{\gamma \in \Lambda^J} T_\gamma\} &= 0.
\end{aligned}$$

So

$$\begin{aligned}
&\{K, T, T\} \\
&= \{K, T_0 \oplus (\oplus_{\alpha \in \Lambda^{-J}} T_\alpha) \oplus (\oplus_{\beta \in \Lambda^J} T_\beta), T_0 \oplus (\oplus_{\alpha \in \Lambda^{-J}} T_\alpha) \oplus (\oplus_{\beta \in \Lambda^J} T_\beta)\} \\
&= \{K, T_0, T_0\} + \{K, T_0, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha\} + \{K, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha, T_0\} + \{K, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha, \oplus_{\gamma \in \Lambda^{-J}} T_\gamma\}.
\end{aligned}$$

It is easy to get $\{K, T_0, T_0\} \subset K$. Let us consider $\{K, T_0, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha\}$ and suppose there exist $\alpha_i \in \Lambda^{J,I}$ and $\alpha \in \Lambda^{-J}$ such that $\{T_{-\alpha_i}, T_0, T_\alpha\} \neq 0$. Since $T_{-\alpha_i} \subset K \subset J$, we get $-\alpha_i + \alpha \in \Lambda^J$. By the root-multiplicativity of T , the symmetries of Λ^J and Λ^{-J} , and the fact $T_{\alpha_i} \subset I$ we obtain $0 \neq \{T_{\alpha_i}, T_0, T_{-\alpha}\} = T_{\alpha_i - \alpha} \subset I$, that is $\alpha_i - \alpha \in \Lambda^{J,I}$. Hence, $-\alpha_i + \alpha \in -\Lambda^{J,I}$ and so $\{T_{-\alpha_i}, T_0, T_\alpha\} = T_{-\alpha_i + \alpha} \subset K$. Similarly, we also get $\{K, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha, T_0\} \subset K$.

At last, we consider $\{K, \oplus_{\alpha \in \Lambda^{-J}} T_\alpha, \oplus_{\gamma \in \Lambda^{-J}} T_\gamma\}$ and suppose there exist $\alpha_i \in \Lambda^{J,I}$, $\alpha \in \Lambda^{-J}$ and $\gamma \in \Lambda^{-J}$ such that $\{T_{-\alpha_i}, T_\alpha, T_\gamma\} \neq 0$. Since $T_{-\alpha_i} \subset K \subset J$, we get $-\alpha_i + \alpha + \gamma \in \Lambda^J$. By the root-multiplicativity of T , the symmetries of Λ^J and Λ^{-J} , and the fact $T_{\alpha_i} \subset I$, one gets $0 \neq \{T_{\alpha_i}, T_{-\alpha}, T_{-\gamma}\} = T_{\alpha_i - \alpha - \gamma} \subset I$, that is $\alpha_i - \alpha - \gamma \in \Lambda^{J,I}$. Hence, $-\alpha_i + \alpha + \gamma \in -\Lambda^{J,I}$ and so $\{T_{-\alpha_i}, T_\alpha, T_\gamma\} = T_{-\alpha_i + \alpha + \gamma} \subset K$. Consequently, K is an ideal of T . \square

We introduce the Definition of primeness in the framework of Leibniz triple systems following the same motivation that in the case of simplicity (see Definition 2.7 and the above paragraph).

Definition 3.11. *A Leibniz triple system T is said to be prime if given two ideals I, K of T satisfying $\{I, K, I\} + \{K, I, I\} + \{I, I, K\} = 0$, then either $I \in \{0, J, T\}$ or $K \in \{0, J, T\}$.*

We also note that the above Definition agrees with the Definition of prime Lie triple system, since $J = 0$ in this case.

Under the hypotheses of Proposition 3.10 we have:

Corollary 3.12. *If furthermore T is prime, then any nonzero ideal I of T such that $I \subseteq J$ satisfies $I = J$.*

Proof. Observe that, by Proposition 3.10, we could have $J = I \oplus K$ with I, K ideals of T , being $\{I, K, I\} + \{K, I, I\} + \{I, I, K\} = 0$ as consequence of $I, K \subseteq J$. The primeness of T completes the proof. \square

Proposition 3.13. *Suppose $T = \{T, T, T\}$, $\text{Ann}_{\text{Lie}}(T) = 0$, T is root-multiplicative and for any $\alpha \in \Lambda^1$, we have $\dim L_\alpha^0 = 1$. If $\Lambda^{\neg J}$ has all of its roots $\neg J$ -connected and $\{T_0, T_\alpha, T_\beta\} = \{T_\alpha, T_0, T_\beta\} = \{T_\alpha, T_\beta, T_0\} = 0$, for $\alpha, \beta \in \Lambda^{\neg J}$, then any ideal I of T such that $I \not\subseteq J$ satisfies $I = T$.*

Proof. Taking into account Lemma 3.7 and Proposition 3.9 we just have to study the case in which

$$I = (I \cap T_0) \oplus (\oplus_{\beta_j \in \Lambda^{J, I}} T_{\beta_j}),$$

where $I \cap T_0 \neq 0$. But this possibility never happens. Indeed, observe that

$$\{I \cap T_0, T_0, T_0\} + \{T_0, I \cap T_0, T_0\} + \{T_0, T_0, I \cap T_0\} = 0.$$

We also have

$$\{I \cap T_0, \oplus_{\alpha \in \Lambda^{\neg J}} T_\alpha, T_0\} + \{\oplus_{\alpha \in \Lambda^{\neg J}} T_\alpha, I \cap T_0, T_0\} + \{\oplus_{\alpha \in \Lambda^{\neg J}} T_\alpha, T_0, I \cap T_0\} \subset I \cap \oplus_{\alpha \in \Lambda^{\neg J}} T_\alpha = 0$$

and

$$\{I \cap T_0, T_0, \oplus_{\alpha \in \Lambda^{\neg J}} T_\alpha\} + \{T_0, I \cap T_0, \oplus_{\alpha \in \Lambda^{\neg J}} T_\alpha\} + \{T_0, \oplus_{\alpha \in \Lambda^{\neg J}} T_\alpha, I \cap T_0\} \subset I \cap \oplus_{\alpha \in \Lambda^{\neg J}} T_\alpha = 0.$$

And known condition gives us

$$\begin{aligned} & \{I \cap T_0, \oplus_{\alpha \in \Lambda^{\neg J}} T_\alpha, \oplus_{\beta \in \Lambda^{\neg J}} T_\beta\} + \{\oplus_{\alpha \in \Lambda^{\neg J}} T_\alpha, I \cap T_0, \oplus_{\beta \in \Lambda^{\neg J}} T_\beta\} + \{\oplus_{\alpha \in \Lambda^{\neg J}} T_\alpha, \oplus_{\beta \in \Lambda^{\neg J}} T_\beta, I \cap T_0\} \\ &= 0. \end{aligned}$$

That is, we get $I \cap T_0 \subset \text{Ann}_{\text{Lie}}(T) = 0$, a contradiction. Proposition 3.9 completes the proof. \square

Given any $\alpha \in \Lambda^\gamma$, $\gamma \in \{J, \neg J\}$, we denote by

$$\Lambda_\alpha^\gamma := \{\beta \in \Lambda^\gamma : \beta \sim_{\neg J} \alpha\}.$$

For $\alpha \in \Lambda^\gamma$, we write $T_{0, \Lambda_\alpha^\gamma} := \text{span}_{\mathbb{K}}\{\{T_\tau, T_\beta, T_\gamma\} : \tau + \beta + \gamma = 0; \tau, \beta, \gamma \in \Lambda_\alpha^\gamma\} \subset T_0$, and $V_{\Lambda_\alpha^\gamma} := \oplus_{\beta \in \Lambda_\alpha^\gamma} T_\beta$. We denote by $T_{\Lambda_\alpha^\gamma}$ the following subspace of T , $T_{\Lambda_\alpha^\gamma} := T_{0, \Lambda_\alpha^\gamma} \oplus V_{\Lambda_\alpha^\gamma}$.

Lemma 3.14. *If $T = \{T, T, T\}$, then $T_{\Lambda_\alpha^J}$ is an ideal of T for any $\alpha \in \Lambda^J$.*

Proof. We have $T_{0,\Lambda_\alpha^J} = 0$ and so

$$T_{\Lambda_\alpha^J} = \oplus_{\beta \in \Lambda_\alpha^J} T_\beta.$$

Using Proposition 2.6 (1) and (3.12), we have

$$\begin{aligned} & \{T_{\Lambda_\alpha^J}, T, T\} + \{T, T_{\Lambda_\alpha^J}, T\} + \{T, T, T_{\Lambda_\alpha^J}\} \\ &= \{T_{\Lambda_\alpha^J}, T, T\} + 0 + 0 \\ &= \{T_{\Lambda_\alpha^J}, T, T\} \\ &= \{T_{\Lambda_\alpha^J}, T_0 \oplus (\oplus_{\alpha \in \Lambda^{-J}} T_\alpha) \oplus (\oplus_{\beta \in \Lambda^J} T_\beta), T_0 \oplus (\oplus_{\alpha \in \Lambda^{-J}} T_\alpha) \oplus (\oplus_{\beta \in \Lambda^J} T_\beta)\} \\ &= \{T_{\Lambda_\alpha^J}, T_0, T_0\} + \{T_{\Lambda_\alpha^J}, \oplus_{\gamma \in \Lambda^{-J}} T_\gamma, T_0\} + \{T_{\Lambda_\alpha^J}, T_0, \oplus_{\gamma \in \Lambda^{-J}} T_\gamma\} + \{T_{\Lambda_\alpha^J}, \oplus_{\beta \in \Lambda^{-J}} T_\beta, \oplus_{\gamma \in \Lambda^{-J}} T_\gamma\}. \end{aligned}$$

It is easy to see $\{T_{\Lambda_\alpha^J}, T_0, T_0\} \subset T_{\Lambda_\alpha^J}$. Next we will assert $\{T_{\Lambda_\alpha^J}, \oplus_{\gamma \in \Lambda^{-J}} T_\gamma, T_0\} \subset T_{\Lambda_\alpha^J}$. Indeed, given any $m \in \Lambda_\alpha^J$, $\gamma \in \Lambda^{-J}$ such that $\{T_m, T_\gamma, T_0\} \neq 0$. By $m + \gamma \neq 0$, we have $m + \gamma \in \Lambda^J$ and so $\{m, \gamma, 0\}$ is a $\neg J$ -connection from m to $m + \gamma$. By the symmetry and transitivity of $\sim_{\neg J}$ in Λ^J , we get $m + \gamma \in \Lambda_\alpha^J$. Hence $\{T_{\Lambda_\alpha^J}, \oplus_{\gamma \in \Lambda^{-J}} T_\gamma, T_0\} \subset T_{\Lambda_\alpha^J}$. Similarly, we also get $\{T_{\Lambda_\alpha^J}, T_0, \oplus_{\gamma \in \Lambda^{-J}} T_\gamma\} \subset T_{\Lambda_\alpha^J}$.

At last, we will assert $\{T_{\Lambda_\alpha^J}, \oplus_{\beta \in \Lambda^{-J}} T_\beta, \oplus_{\gamma \in \Lambda^{-J}} T_\gamma\} \subset T_{\Lambda_\alpha^J}$. Indeed, given any $m \in \Lambda_\alpha^J$, $\beta, \gamma \in \Lambda^{-J}$ such that $\{T_m, T_\beta, T_\gamma\} \neq 0$. By $m + \beta + \gamma \neq 0$, we have $m + \beta + \gamma \in \Lambda^J$ and so $\{m, \beta, \gamma\}$ is a $\neg J$ -connection from m to $m + \beta + \gamma$. By the symmetry and transitivity of $\sim_{\neg J}$ in Λ^J , we get $m + \beta + \gamma \in \Lambda_\alpha^J$. Hence $\{T_{\Lambda_\alpha^J}, \oplus_{\beta \in \Lambda^{-J}} T_\beta, \oplus_{\gamma \in \Lambda^{-J}} T_\gamma\} \subset T_{\Lambda_\alpha^J}$.

Consequently, $\{T_{\Lambda_\alpha^J}, T, T\} + \{T, T_{\Lambda_\alpha^J}, T\} + \{T, T, T_{\Lambda_\alpha^J}\} \subset T_{\Lambda_\alpha^J}$. So $T_{\Lambda_\alpha^J}$ is an ideal of T for any $\alpha \in \Lambda^J$. \square

Theorem 3.15. Suppose $T = \{T, T, T\}$, $\text{Ann}_{\text{Lie}}(T) = 0$, T is root-multiplicative and for any $\alpha \in \Lambda^1$, we have $\dim L_\alpha^0 = 1$. If Λ^J, Λ^{-J} are symmetric and $\{T_0, T_\alpha, T_\beta\} = \{T_\alpha, T_0, T_\beta\} = \{T_\alpha, T_\beta, T_0\} = 0$, for $\alpha, \beta \in \Lambda^{-J}$, then T is simple if and only if it is prime and Λ^J, Λ^{-J} have all of their roots $\neg J$ -connected.

Proof. Suppose T is simple. If $\Lambda^J \neq \emptyset$ and we take $\alpha \in \Lambda^J$. Lemma 3.14 gives us $T_{\Lambda_\alpha^J}$ is a nonzero ideal of T and so, (by simplicity), $T_{\Lambda_\alpha^J} = J = \oplus_{\beta \in \Lambda^J} T_\beta$ (see (3.10) and (3.20)). Hence, $\Lambda_\alpha^J = \Lambda^J$ and consequently Λ^J has all of its roots $\neg J$ -connected.

Consider now any $\gamma \in \Lambda^{-J}$ and the subspace $T_{\Lambda_\gamma^{-J}}$. Let us denote by $I(T_{\Lambda_\gamma^{-J}})$ the ideal of T generated by $T_{\Lambda_\gamma^{-J}}$. We observe that the fact J is an ideal of T and we assert that $I(T_{\Lambda_\gamma^{-J}}) \cap (\oplus_{\delta \in \Lambda^{-J}} T_\delta)$ is contained in the linear span of the set

$$\begin{aligned} & \left\{ \{ \cdots \{v_{\gamma'}, v_{\alpha_1}, v_{\alpha_2}\}, \cdots \}, v_{\alpha_{2n}}, v_{\alpha_{2n+1}} \}; \{v_{\alpha_{2n+1}}, v_{\alpha_{2n}}, \{ \cdots \{v_{\alpha_2}, v_{\alpha_1}, v_{\gamma'}\}, \cdots \} \}; \right. \\ & \{ \{ \cdots \{v_{\alpha_1}, v_{\alpha_2}, v_{\gamma'}\}, \cdots \}, v_{\alpha_{2n}}, v_{\alpha_{2n+1}} \}; \{v_{\alpha_{2n+1}}, v_{\alpha_{2n}}, \{ \cdots \{v_{\gamma'}, v_{\alpha_2}, v_{\alpha_1}\}, \cdots \} \}; \\ & \left. \{ \{ \cdots \{v_{\alpha_1}, v_{\gamma'}, v_{\alpha_2}\}, \cdots \}, v_{\alpha_{2n}}, v_{\alpha_{2n+1}} \}; \{v_{\alpha_{2n+1}}, v_{\alpha_{2n}}, \{ \cdots \{v_{\alpha_2}, v_{\gamma'}, v_{\alpha_1}\}, \cdots \} \}; \right\} \\ & \text{where } 0 \neq v_{\gamma'} \in T_{\Lambda_\gamma^{-J}}, 0 \neq v_{\alpha_i} \in T_{\alpha_i}, \alpha_i \in \Lambda^{-J} \text{ and } n \in \mathbb{N} \}. \end{aligned}$$

By simplicity $I(T_{\Lambda_\gamma^{-J}}) = T$. From here, given any $\delta \in \Lambda^{-J}$, the above observation gives us that we can write $\delta = \gamma' + \alpha_1 + \cdots + \alpha_{2n+1}$ for $\gamma' \in \Lambda_\gamma^{-J}$, $\alpha_i \in \Lambda^{-J}$ and being the

partial sums nonzero. Hence $\{\gamma', \alpha_1, \dots, \alpha_{2n+1}\}$ is a $\neg J$ -connection from γ' to δ . By the symmetry and transitivity of $\sim_{\neg J}$ in $\Lambda^{\neg J}$, we deduce γ is $\neg J$ -connected to any $\delta \in \Lambda^{\neg J}$. Consequently, Proposition 3.5 gives us that $\Lambda^{\neg J}$ has all of its roots $\neg J$ -connected. Finally, since T is simple then is prime.

The converse is consequence of Corollary 3.12 and Proposition 3.13. \square

References

- [1] M. Bremner, J. Sánchez-Ortega, (2014), Leibniz triple systems. Commun. Contemp. Math. 16(1), 1350051, 19 pp.
- [2] P. Kolesnikov, (2008), Varieties of dialgebras, and conformal algebras. (Russian) Sibirsk. Mat. Zh. 49(2), 322-339; translation in Sib. Math. J. 49(2), 257-272.
- [3] Yao Ma, Liangyun Chen, (2014), Some structures of Leibniz triple systems. arXiv:1407.3978.
- [4] A. J. Calderón, M. F. Piulestán, (2010), On split Lie triple systems II. Proc. Indian Acad. Sci. (Math. Sci.) 120 (2), 185-198.
- [5] A. J. Calderón, J. M. Sánchez, (2012), On split Leibniz algebras. Linear Algebra Appl. 436 (6), 1648-1660.
- [6] A. J. Calderón, (2008), On split Lie algebras with symmetric root systems. Proc. Indian Acad. Sci. (Math. Sci.) 118, 351-356.
- [7] A. J. Calderón, (2009), On split Lie triple systems. Proc. Indian Acad. Sci. (Math. Sci.) 119 (2), 165-177.
- [8] A. J. Calderón, (2009), On simple split Lie triple systems. Algebr. Represent. Theory 12, 401-415.